Modelling and Information Entropy of Design Spaces

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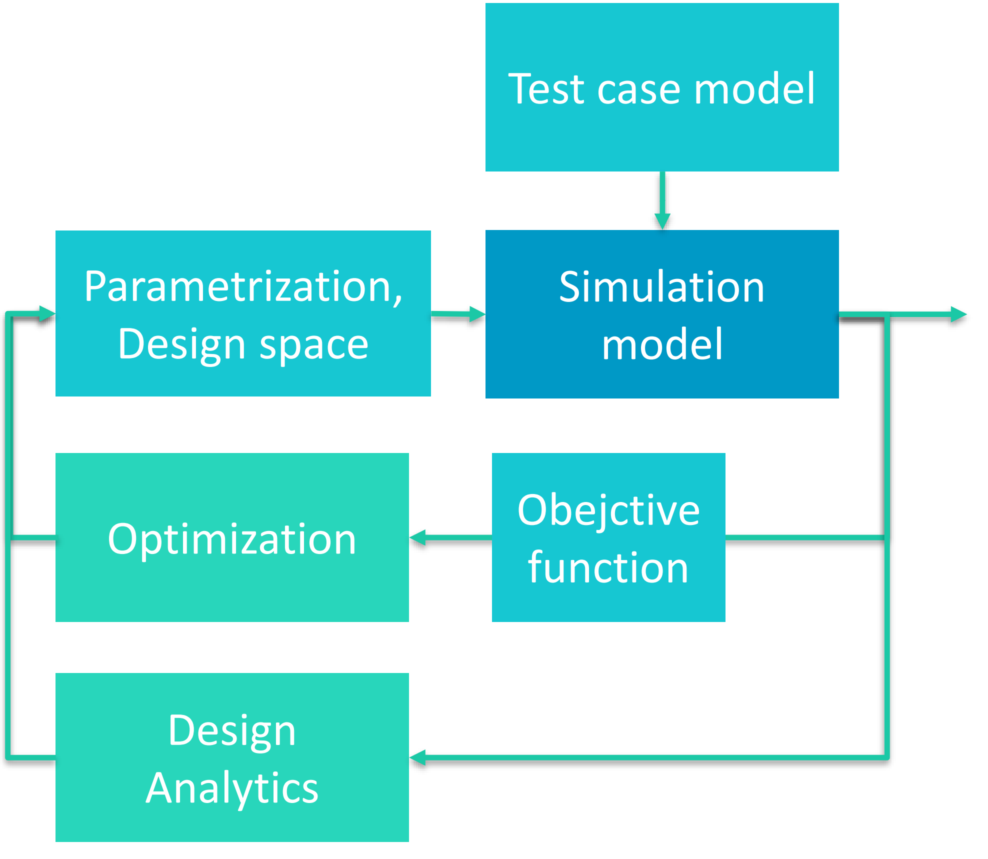
Extended System Simulation

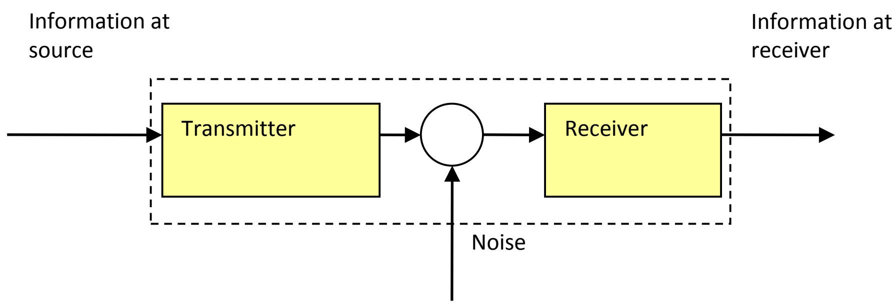
- Connectivity, co-simulation, multi-core, FMU etc.
- Simulation based optimization
- Design analytics
  - I.e. sensitivity analysis, correlation analysis, robustness, complexity metrics, etc.
  - Methods for experimental validation
- Parametrization for design.
  - Analytic parametrization, and reduction
- Test case modelling and management

Extended System Simulation



Information Theory

- Claude Shannons ground breaking work.



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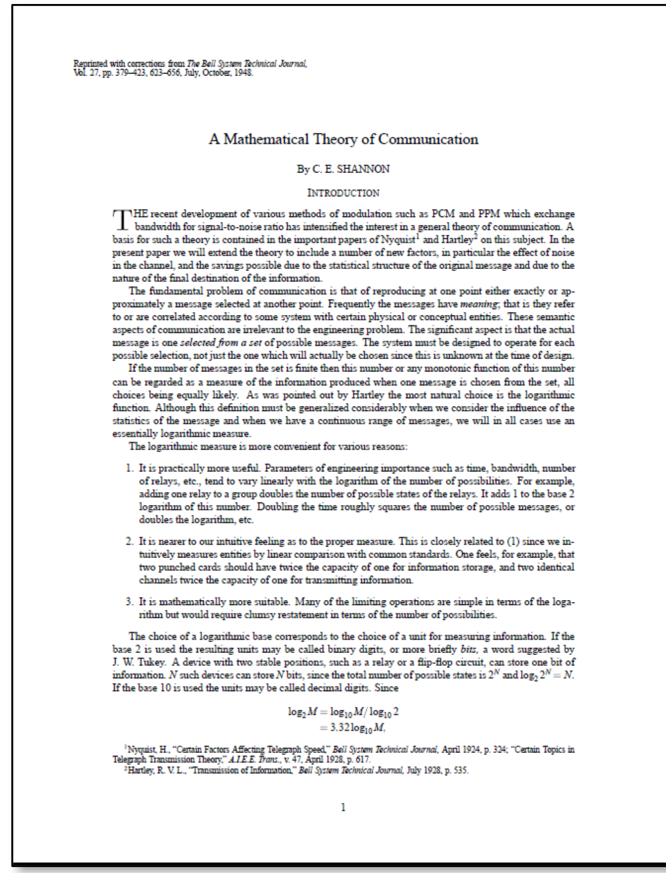

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Vol. 27, pp. 379–423, 623–656, July, October, 1948.

A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded as a measure of the information produced when one message is chosen from the set, all choices being equally likely. As was pointed out by Hartley the most natural choice is the logarithmic function. Although this definition must be generalized considerably when we consider the influence of the statistics of the message and when we have a continuous range of messages, we will in all cases use an essentially logarithmic measure.

The logarithmic measure is more convenient for various reasons:

1. It is practically more useful. Parameters of engineering importance such as time, bandwidth, number of relays, etc., tend to vary linearly with the logarithm of the number of possibilities. For example, adding one relay to a group doubles the number of possible states of the relays. It adds 1 to the base 2 logarithm of this number. Doubling the time roughly squares the number of possible messages, or doubles the logarithm, etc.

2. It is nearer to our intuitive feeling as to the proper measure. This is closely related to (1) since we intuitively measures entities by linear comparison with common standards. One feels, for example, that two punched cards should have twice the capacity of one for information storage, and two identical channels twice the capacity of one for transmitting information.

3. It is mathematically more suitable. Many of the limiting operations are simple in terms of the logarithm but would require clumsy restatement in terms of the number of possibilities.

The choice of a logarithmic base corresponds to the choice of a unit for measuring information. If the base 2 is used the resulting units may be called binary digits, or more briefly *bits*, a word suggested by J. W. Tukey. A device with two stable positions, such as a relay or a flip-flop circuit, can store one bit of information. N such devices can store N bits, since the total number of possible states is 2^N and $\log_2 2^N = N$. If the base 10 is used the units may be called decimal digits. Since

$$\log_2 M = \log_{10} M / \log_{10} 2$$
$$= 3.32 \log_{10} M,$$

[1]Nyquist, H., "Certain Factors Affecting Telegraph Speed," *Bell System Technical Journal,* April 1924, p. 324; "Certain Topics in Telegraph Transmission Theory," *A.I.E.E. Trans.*, v. 47, April 1928, p. 617.
[2]Hartley, R. V. L., "Transmission of Information," *Bell System Technical Journal,* July 1928, p. 535.

Applications

- Product Platforms
- Complexity of Computer Programs
- Logic Hardware Design
- Human factors
- Search Theory
- Axiomatic Design
- System Design
- Optimizaton

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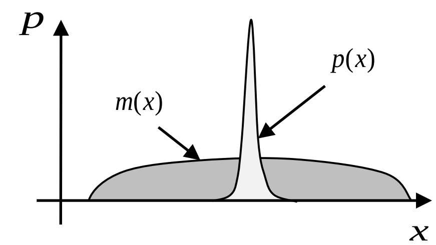
Amount of Information content (Information Entropy)

The differential information entropy for continuous signals, defined by Shannon [1] as:

$$H = -\int_{-\infty}^{\infty} p(x)\log_2(p(x))dx$$

Kullback-Leibler divergence

$$H_{rel} = -\int_{-\infty}^{\infty} p(x)\log_2\left(\frac{p(x)}{m(x)}\right)dx$$



Generalized

$$H_{rel} = -\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(x_1\ldots x_n)\log_2\left(\frac{p(x_1\ldots x_n)}{m(x_1\ldots x_n)}\right)dx_1\cdots dx_n$$

Amount of Information content (Information Entropy)

If the distribution *m*(*x*) is a rectangular distribution in the bounded interval.

$$I_x = H_{rel}(x) = -\int_{x_{\min}}^{x_{\max}} p(x)\log_2\left(p(x)x_R\right)dx$$

Generalized

$$I_x = -\int_{x_{1,\min}}^{x_{1,\max}} \cdots \int_{x_{n,\min}}^{x_{n,\max}} p(x_1 \ldots x_n)\log_2(p(x_1,\ldots x_n)x_{R1} \cdots x_{Rn})dx_1 \cdots dx_n$$

More compact

$$I_x = \int_D p(\mathbf{x})\log_2(p(\mathbf{x})D)dD$$

Amount of Information content (Information Entropy)

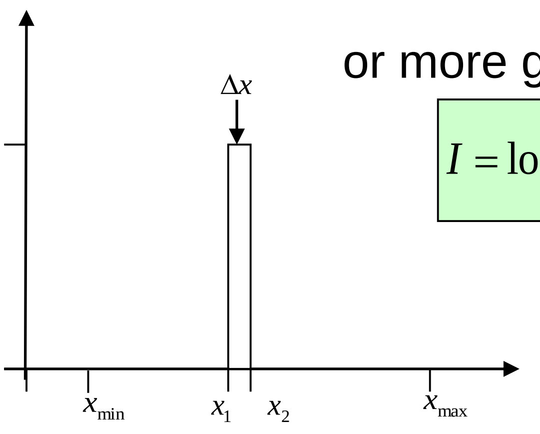
The information content I of a variable (in bits).

$$I = -\int_{x_{\min}}^{x_{\max}} p(x)\log_2\left(p(x)x_R\right)dx \qquad x_R = x_{\max} - x_{\min}$$

If the range x_r is divided in equal parts Δx the amount of information is:

$$I = \log_2 \frac{x_r}{\Delta x} = \log_2 \frac{1}{\delta_x}$$

Here Δx is the tolerance in x.



or more general:

$$I = \log_2 \frac{S_0}{s}$$

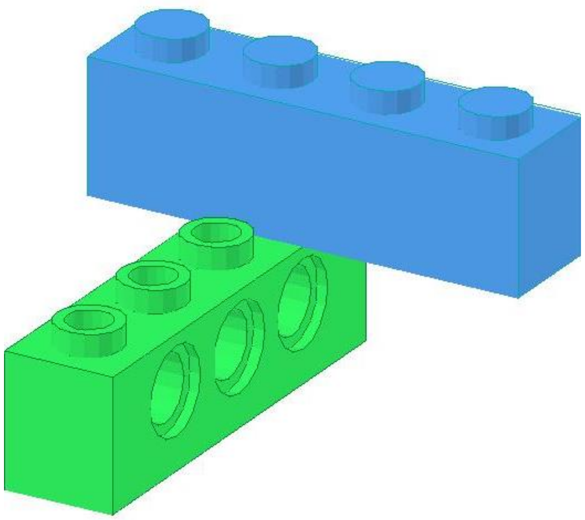
The choice of logarithm as a base (Shannon)

- It is practically more useful. Parameters of engineering importance such as time, bandwidth, number of relays, etc., tend to vary linearly with the logarithm of the number of possibilities. For example, adding one relay to a group doubles the number of possible states of the relays. It adds 1 to the base 2 logarithm of this number. Doubling the time roughly squares the number of possible messages, or doubles the logarithm, etc.
- It is nearer to our intuitive feeling as to the proper measure. This is closely related to (1) since we intuitively measures entities by linear comparison with common standards. One feels, for example, that two punched cards should have twice the capacity of one for information storage, and two identical channels twice the capacity of one for transmitting information.
- It is mathematically more suitable. Many of the limiting operations are simple in terms of the logarithm but would require clumsy restatement in terms of the number of possibilities.

Design Information Entropy is a Measure of the Size of the Design Space

Lego example

- The design space of a set of Lego bricks represents all combinations of arranging these bricks.
- With a set of only two bricks with four knobs on each there are 51 discrete possible arrangements
- Two of these represents picking only one brick. And one state is to pick no one.
- The 51 different configuration (states) means that the amount of information needed to specify a particular design is:



$$I_x = \log_2 n_{Dstate} = \log_2 51 = 5.7\text{bits}$$

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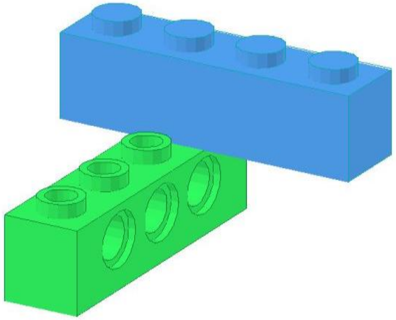

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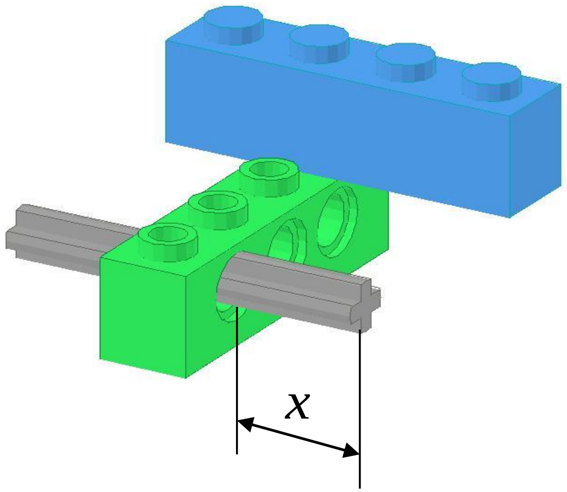
Design Space with Both Continuous and Discrete Variables

- The position of the inserted axis represents a continuous variables
- The information entropy associated with that is dependent on the accuracy with which it is specified.

$$I_x = \log_2 n_{Dstates} + \log n_{Cstates} + \log_2 \frac{x_R}{\Delta x}$$

- The axis can be in three position and If the position of the axis within one hole is specified within 10% the total information entropy is:

$$I_x = \log_2(51+3) + \log_2 \frac{1}{0.1} = 8.2\text{bits}$$

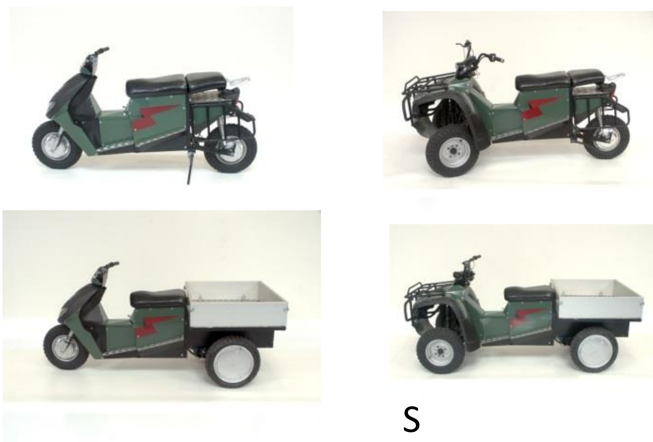


Information entropy in modular design

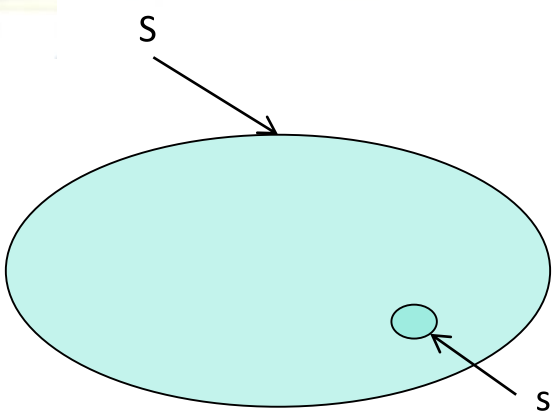
Valid design	Unvalid design
Valid design	Chosen design

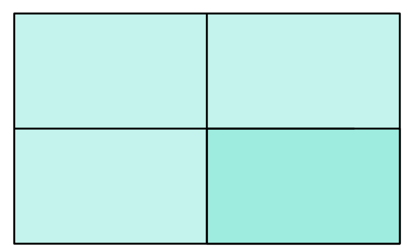
Design information entropy can be used as a measure of quality of product platforms for modular design. A good product platform should have little “waste” of design space.

Design Information



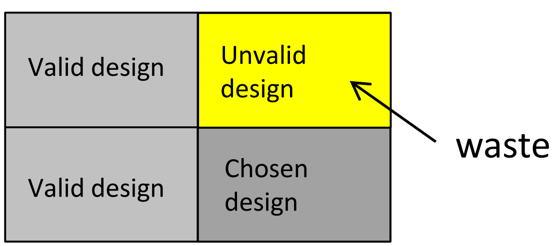
- Design information entropy
- The amount of bits needed to specify a design within a design space.
- To specify one designs of the four takes:





$$H_x = \log_2 \frac{S}{s} = \log_2 n_s = \log_2 (2 \times 1 \times 2) = 2 \text{ bits}$$

Design Information



- To specify one designs of the four takes:

$$H_x = \log_2 \frac{S}{s} = \log_2 n_s = \log_2 (2 \times 1 \times 2) = 2 \text{ bits}$$

- The entropy of the constraint design space is

$$H_c = \log_2 n_v = \log_2 3 = 1.58$$

- Wasted design information entropy can be formulated as:

$$H_w = H_x - H_c = -\log_2 \frac{S_x / s}{S_c / s} = -\log_2 \frac{S_x}{S_c} = 0.42$$

Morphological Matrix for concept selection

$$N_s = \prod_{i=1}^{n_f} n_{m,i}$$

$$N_s = \prod_{i=1}^{4} 3 = 3^4 = 81$$

$$H_x = \log_2 N_s = \log_2 81 = 6.34 \text{ bits}$$

	m^1	m^2	m^3
f_1	m_1^1	m_1^2	m_1^3
f_2	m_2^1	m_2^2	m_2^3
f_3	m_3^1	m_3^2	m_3^3
f_4	m_4^1	m_4^2	m_4^3

Increasing the number of rows by one:

$$N_s = \prod_{i=1}^{4} 3 = 3^5 = 243$$

$$H_x = \log_2 N_s = \log_2 243 = 7.92 \text{ bits}$$

	m^1	m^2	m^3
f_1	m_1^1	m_1^2	m_1^3
f_2	m_2^1	m_2^2	m_2^3
f_3	m_3^1	m_3^2	m_3^3
f_4	m_4^1	m_4^2	m_4^3
f_5	m_5^1	m_5^2	m_5^3

The entropy increases linearly with number of rows.

Information Entropy gives a measure of complexity more consistent with experience!

Information Entropy of Morphological Matrix

In the general case there can be variable number of elements in each row.

$$N_s = \prod_{i=1}^{4} 4 \times 2 \times 3 \times 2 \times 3 = 144$$

$$H_x = \log_2 N_s = \log_2 144 = 7.16 \text{ bits}$$

	m^1	m^2	m^3	m^4
f_1	m_1^1	m_1^2	m_1^3	m_1^3
f_2	m_2^1	m_2^2		
f_3	m_3^1	m_3^2	m_3^3	
f_4	m_4^1	m_4^2		
f_5	m_5^1	m_5^2	m_5^3	

Information Entropy and Complexity

According to Axiomatic Design the best designs are uncoupled

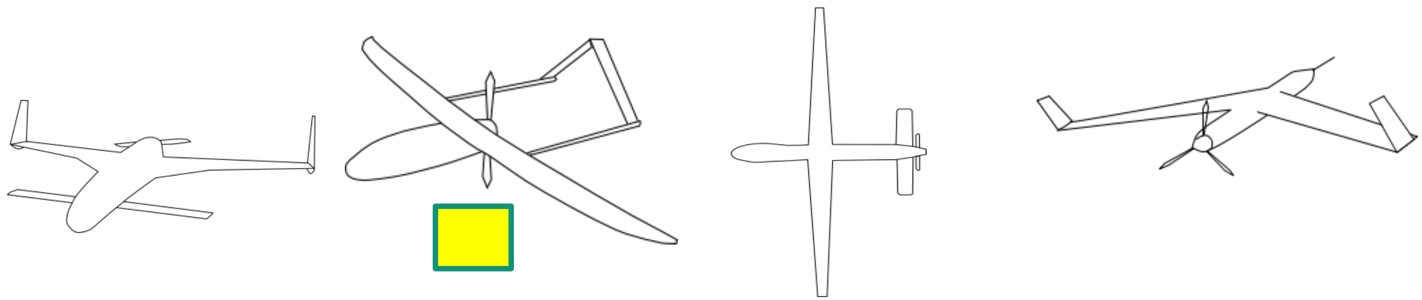
$$\begin{pmatrix} FR_1 \\ FR_2 \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} DP_1 \\ DP_2 \end{pmatrix}$$

Functional requirements (Anatomy)

Design parameters (Architecture)

If this is true. Design decision becomes independent of each other

Example: UAV Aircraft Concept Generation



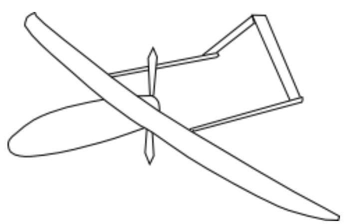
Design elements	Alternative solutions				
Horizontal stabilization	Front (canard)	Aft	Aft tail integrated	Wing integrated	
Vertical stabilization	Central	Wing tip	Aft tail integrated	Upper	Lower
Tail mount	Single fuselage	Twin boom			
Propulsion	Tractor	Pusher			

$$N_s = \prod_{i=1}^{n_f} n_{m,i} \qquad n_s = 4 \times 5 \times 2 \times 2 = 80$$

$$I_x = -\log_2 \frac{1}{N_s} = -\log_2 \frac{1}{80} = 6.32 \text{ bit}$$

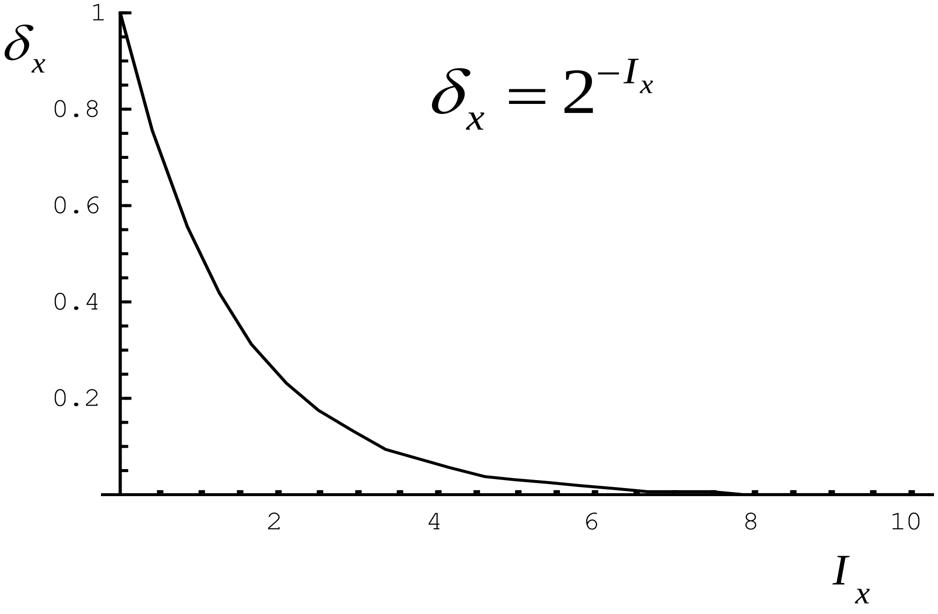
Aircraft Optimization

- For the aircraft example typical design parameters would be;
  - wing span, root cord, tapering, thickness, and sweep, structural weight, fuel weight, engine size, wing position, span of horizontal tail, cruise speed.



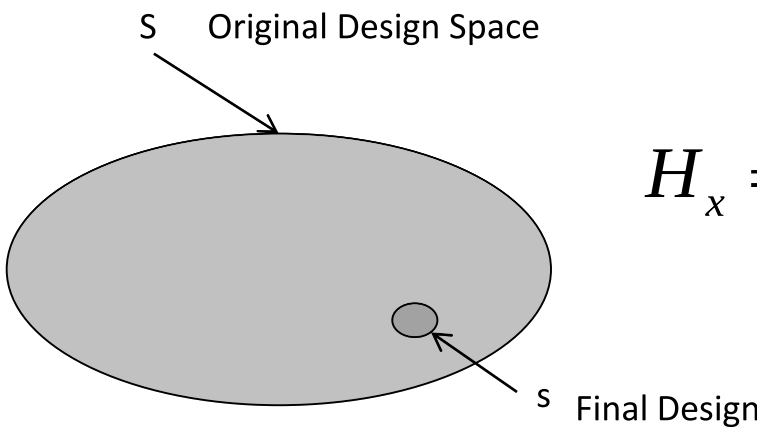
Design uncertainty as a function of design information

$$I_x = \log_2 \frac{x_R}{\Delta x} = \log_2 \delta_x$$



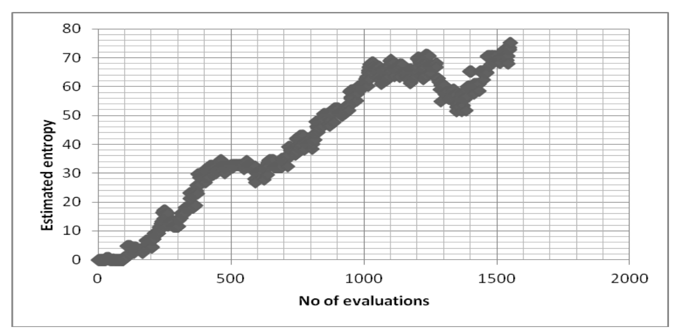
$$\delta_x = 2^{-I_x}$$

Information entropy of design s relative to design space S



$$H_x = \log_2 \frac{S}{s}$$

Information entropy to measure convergence in optimization



Information increase in Optimization

Information entropy is estimated as

$$\hat{H}_x = -n \log_2 \left(\max \left(\delta_{x,i} \right) \right)$$

$$\delta_{x,i} = \frac{x_{i,\max} - x_{i,\min}}{x_{0,i,\max} - x_{0,i,\min}}$$

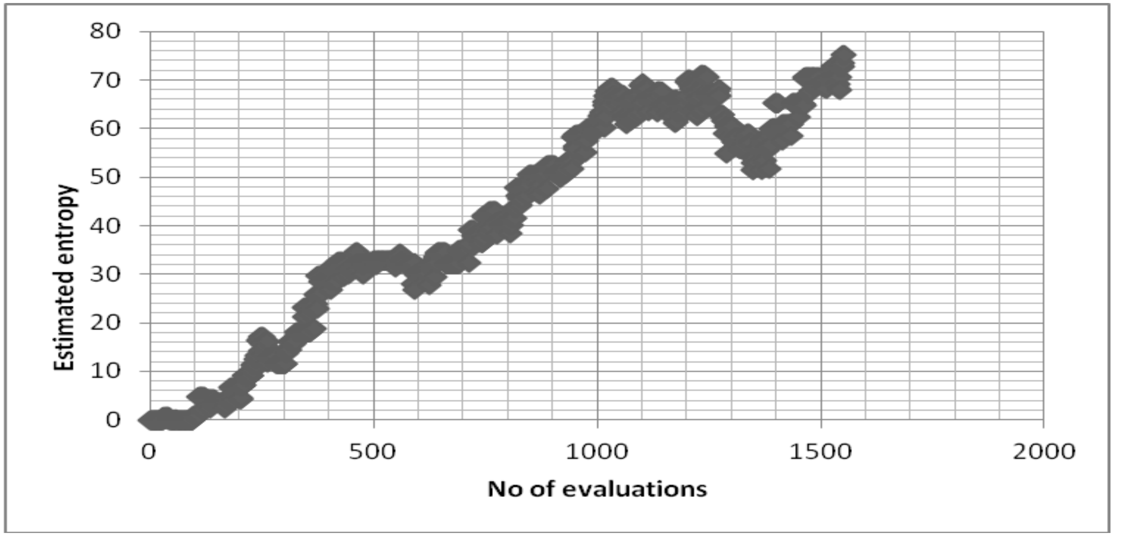


Figure 5. Accumulation of information as a function of number of objective function evaluations

Meta object function

$$I_{tot} = (1-P_{opt})\log_2\left(\frac{1-P_{opt}}{1-\varepsilon_x^n}\right) + P_{opt}\log_2\left(\frac{P_{opt}}{\varepsilon_x^n}\right)$$

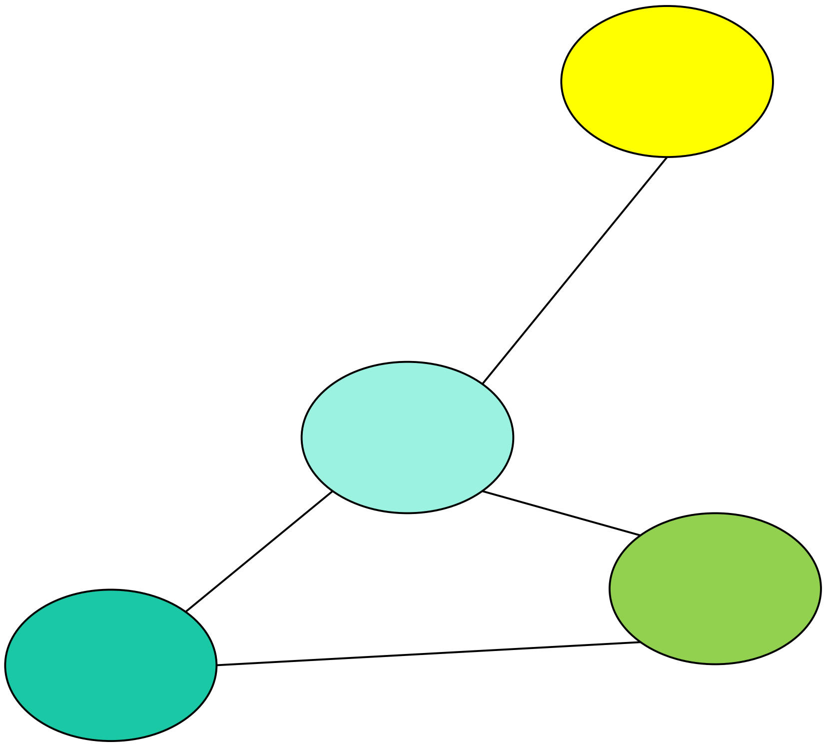
$$\phi^{(2)} = \frac{I_x}{N_m} = \frac{1}{N_m}\left((1-P_{opt})\log_2\left(\frac{1-P_{opt}}{1-\varepsilon_x^n}\right) + P_{opt}\log_2\left(\frac{P_{opt}}{\varepsilon_x^n}\right)\right)$$

I expresses the total uncertainty, representing the sum of

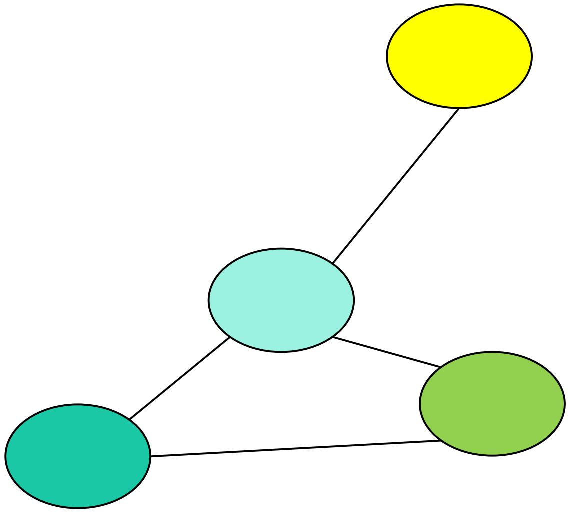
uncertainty in location and

uncertainty of success

Information Entropy in System Design



Information Entropy in System Design



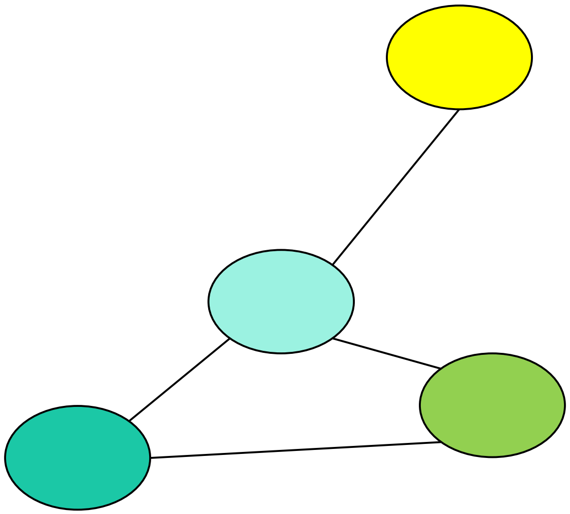
A system configuration is defined by its components and by how they are connected. The design space can this be expressed as:

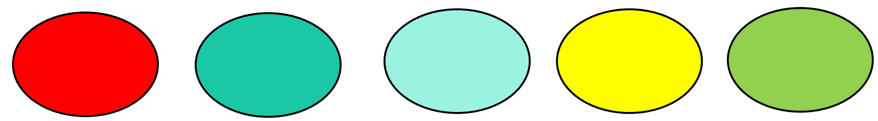
$$N_x = N_s \times N_p \qquad (9.35)$$

Here N_s is the number of possibilities for component selections, and N_p the number of possible ways to connect the components. The corresponding information entropy is:

$$I_x = \log_2 N_x = \log_2 N_s + \log_2 N_p \qquad (9.36)$$

Information Entropy in System Design



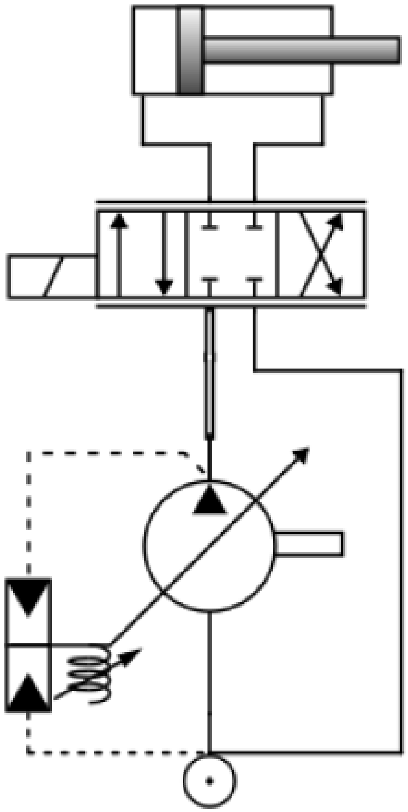


$$n_{s,tot} = 5$$

$$n = 4$$

$$N_s = 5^4 = 625$$

$$I_s = \log_2 N_s = 9.29\text{bit}$$



Information Entropy in System Design

Figure 9.10: System configuration of a simple hydraulic system.

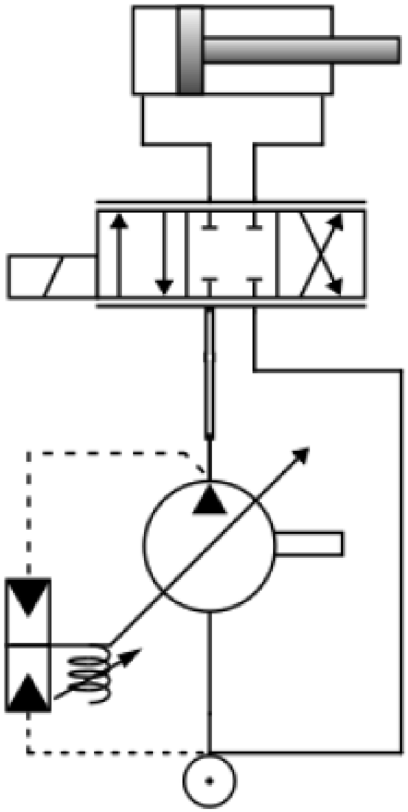
Assuming apriori information that the system should contain one cylinder one pump and one tank and a library with variants of these

$$N_s = N_{cyl} \times N_{valve} \times N_{pump} \times N_{tank} = 6 \times 27 \times 6 \times 1 = 972 \quad (9.43)$$

and hence

$$I_s = \log_2 N_s = \log_2 972 = 9.92 \quad (9.44)$$

Connectivity



System components	Connectors	A	B	A	B	P	R	P	R	R
Piston	A									
	B									
Servo valve	A	1								
	B		1							
	P									
	R									
Pump	P					1				
	R									
Tank	R						1		1	

$$I_p = \frac{n_p \times n_p}{2} - n_p \tag{9.45}$$

Here n_p is the total number of ports in the system. For this example it is:

$$I_p = \frac{10 \times 10}{2} - 10 = 40 \tag{9.46}$$

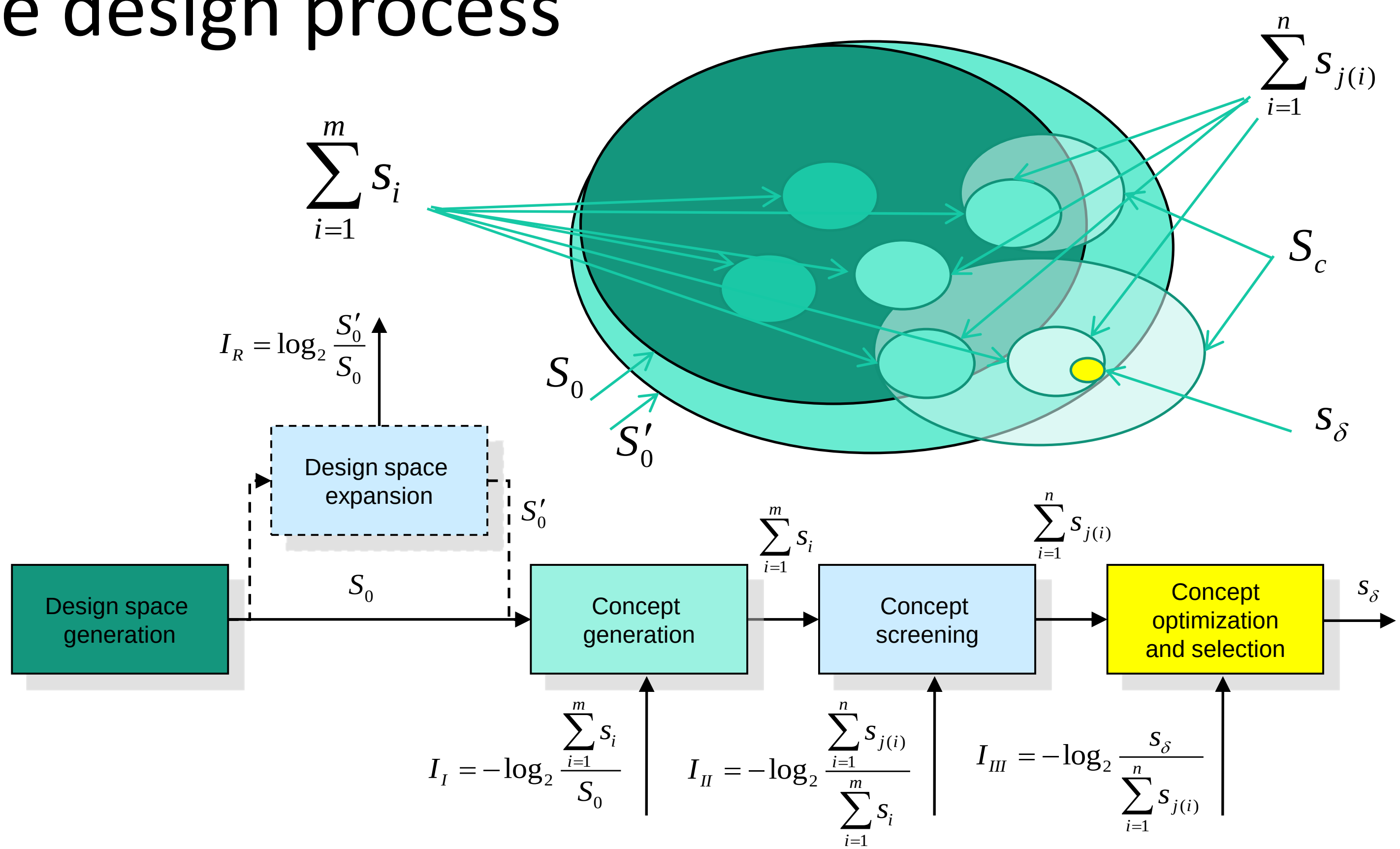
The total amount of information is:

$$I_x = I_s + I_p \tag{9.47}$$

$$I_x = 9.92 + 40 = 49.92\text{bits} \tag{9.48}$$

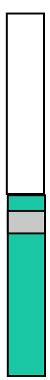
Figure 9.11: Connectivity matrix of a hydraulic servo.

Growth of design information entropy during the design process

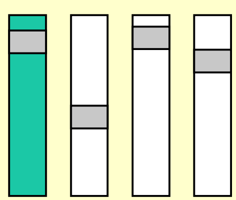


Design space expansion

- The design information entropy can be increased in two ways
  - Refinement
  - Design space expansion
- Design space can be increased in several ways like:
  - Adding more bricks
  - Adding other types of bricks
  - Releasing more design parameters in a design



$$I'_x = \log_2 \frac{x'_R}{\Delta x} = \log_2 \left(\frac{n \times x_R}{\Delta x} \right) = \log_2 n + \log_2 \frac{x_R}{\Delta x} = \log_2 n + I_x$$



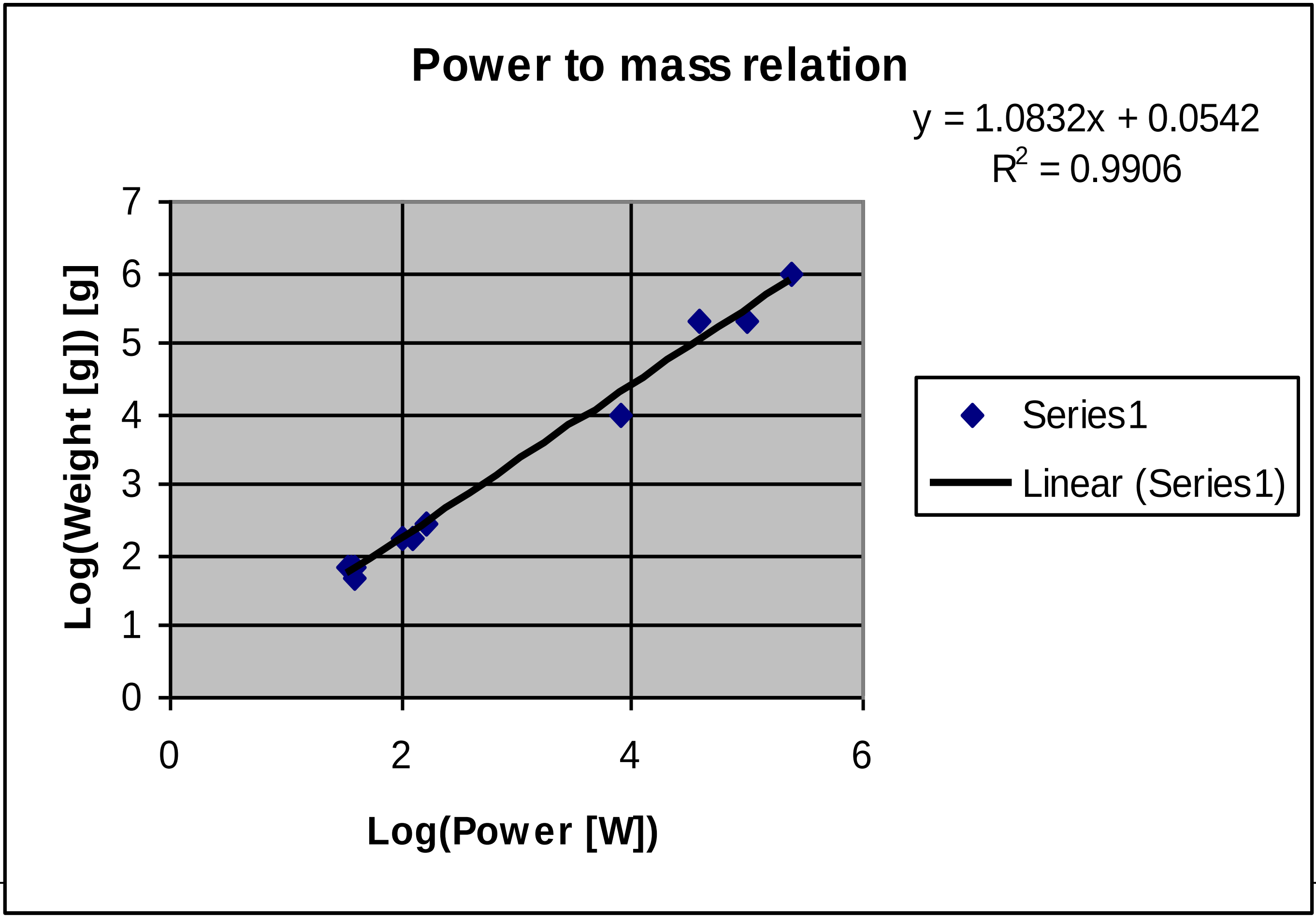
$$I'_x = \log_2 \frac{x'_R}{\Delta x'} = \log_2 \left(\frac{x_R}{\Delta x} \right)^n = n \log_2 \frac{x_R}{\Delta x} = nI_x$$

Design Space Generation and Parameter Reduction

Example: Electric Motor Data

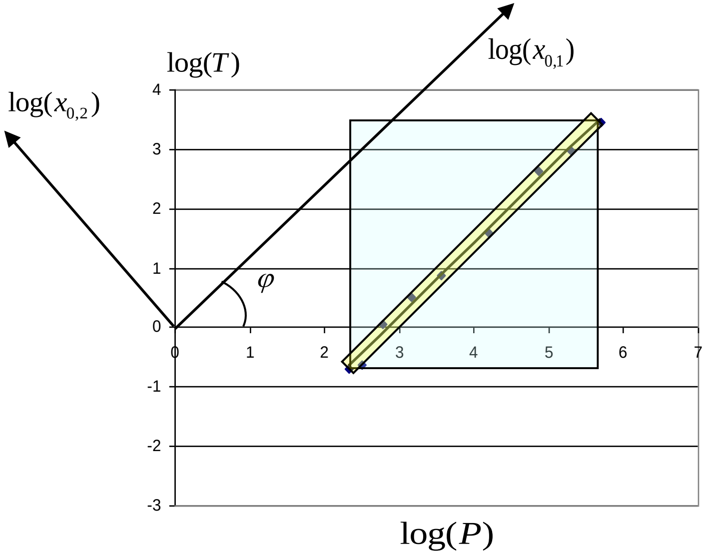
Voltage [V]	max power [W]	speed at load [rad/s]	max torque [Nm]	volume [cm3]	mass [kg]	power intensity k_p [kW/kg]	mean pressure (torque density) p_m [bar]	power density ρ_p [W/cm3]	torque intensity k_T [Nm/kg]
8.4	210	1068	0.20	55	0.1716	1.22	0.04	3.84	1.15
8	320	1378	0.23	54	0.29	1.10	0.04	5.95	0.80
24	609	523	1.164	343	1.10	0.55	0.03	1.78	1.06
24	1440	450	3.2	729	2.4	0.60	0.04	1.98	1.33
24	3580	471	7.6	729	3.9	0.92	0.10	4.91	1.95
50	15992	419	38.20	4539	9.36	1.71	0.08	3.52	4.08
460	73763	175	420.74	23487	215.00	0.34	0.18	3.14	1.96
460	198499	215	922.87	86524	215.00	0.92	0.11	2.29	4.29
460	491751	175	2813.33	165518	907.00	0.54	0.17	2.97	3.10
					Average values	0.88	0.09	3.38	2.19

Power to weight relation (electric motor)

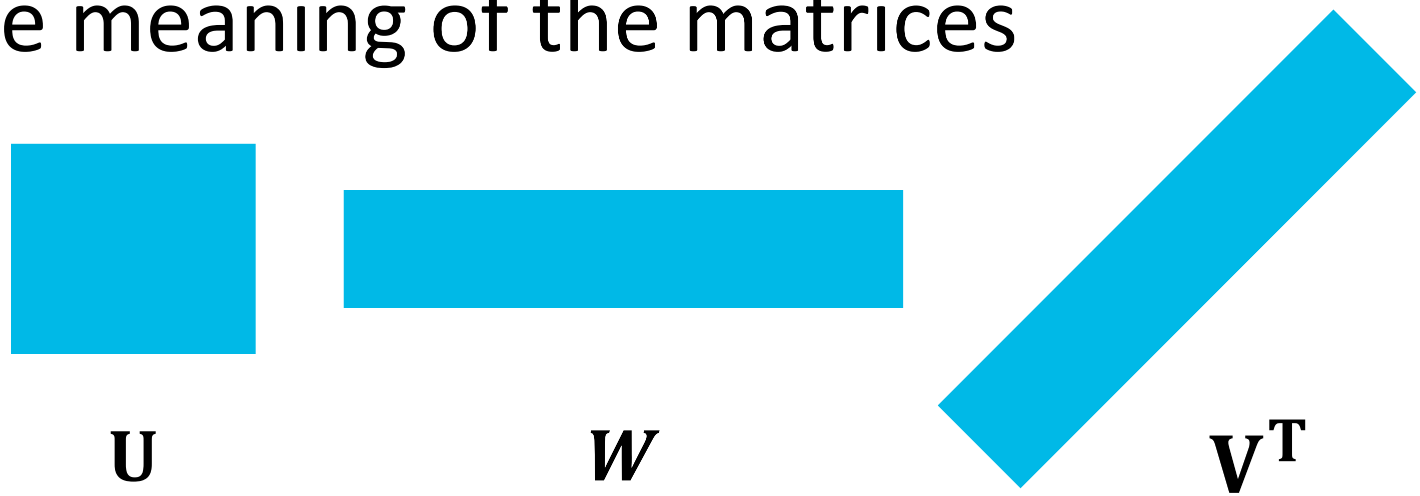


Prinicipal Component Analysis

to minimize waste of design space (using Singular Value Decomposition, SVD)



The meaning of the matrices

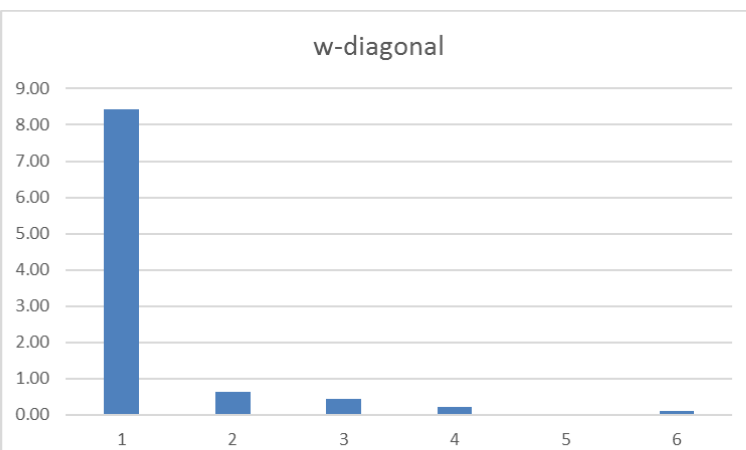


$$\mathbf{X} = \mathbf{U} \times \mathbf{W} \times \mathbf{V}^{\mathrm{T}}$$

Adding logarithmic scaling and removal of offset (set mean to zero) means that limits in U on set to -1 and 1 covers the data set within one standard deviation, and thus results points likely to be feasable.

SVD model of Electric motors

	Reliance K22M13031	Estimate	Adjusted	Result	Average							SVD variables	w-diagonal	residual
Corner power [W]	491751	491744	5.69	1.99	3.70	-1.163	0.130	-0.022	0.007	0.000	-0.001	-1.66	8.43	1.86
Speed at load [rad/s]	175	175	2.24	-0.46	2.70	0.331	0.122	-0.021	-0.031	0.000	0.005	0.39	0.62	0.20
Max Torque [Nm]	2813	2813	3.45	2.44	1.01	-1.494	0.008	-0.001	0.038	0.000	-0.005	-0.56	0.44	0.17
diameter [mm]	446	446	2.65	0.63	2.02	-0.384	0.005	0.064	0.007	0.000	0.028	-0.99	0.21	0.10
volume [cm3]	165518	165516	5.22	2.06	3.16	-1.256	-0.018	0.084	-0.036	0.000	-0.011	-1.01	0.00	0.07
mass [kg]	907	907	2.96	2.21	0.75	-1.298	-0.079	-0.085	-0.025	0.000	0.010	1.05	0.10	0.07

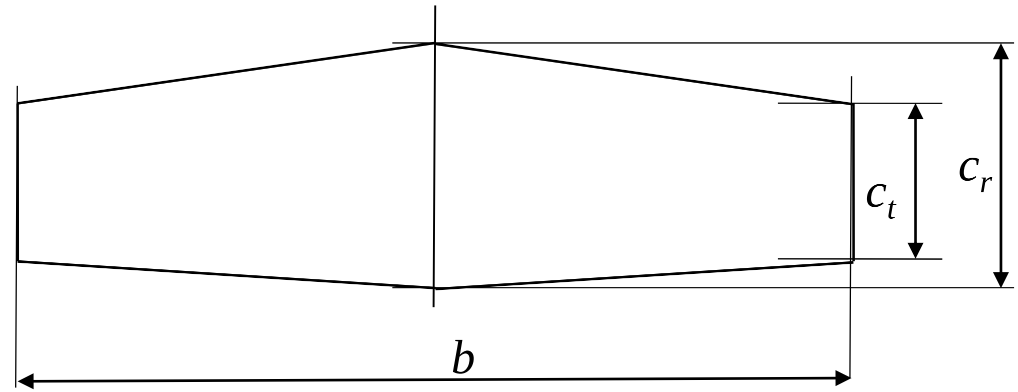


Example of Design Space for Parametrization: Aircraft Wing Planform

$$S = \frac{b}{2}(c_r + c_t)$$

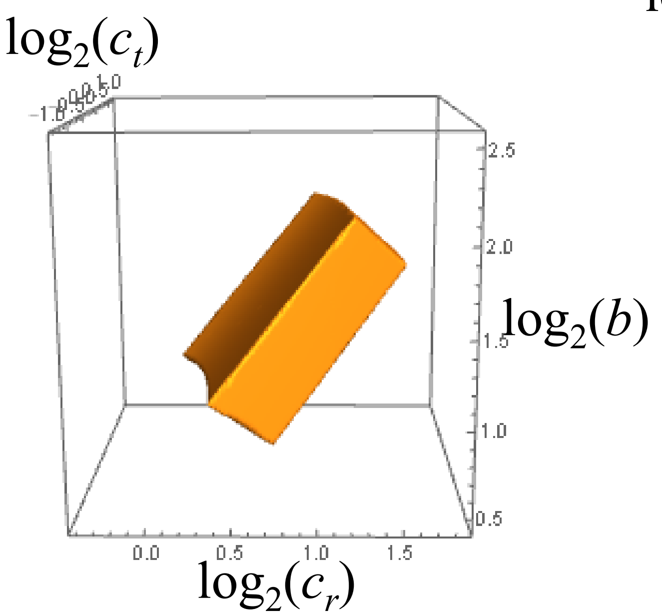
$$AR = \frac{b^2}{S}$$

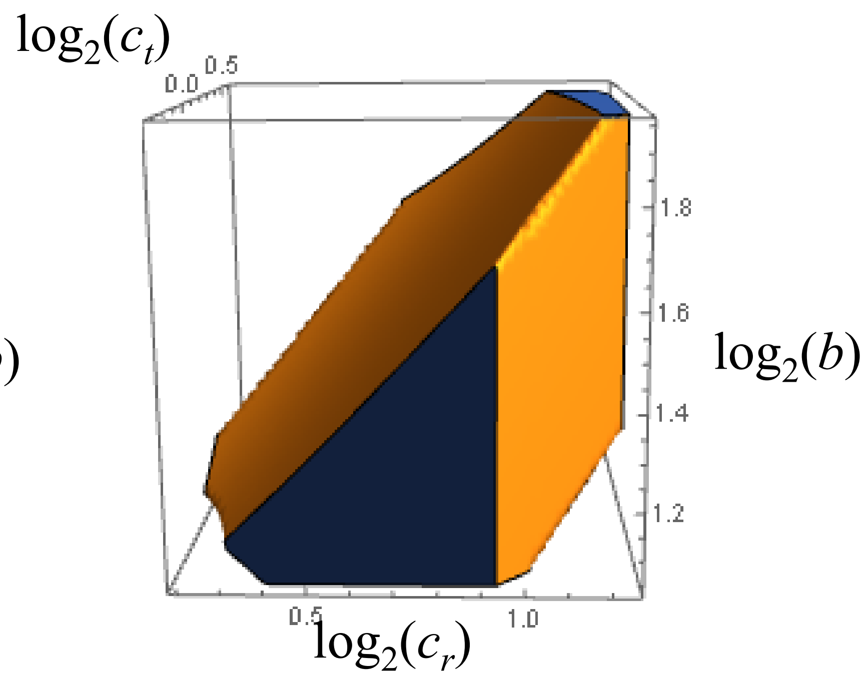
$$c_t = c_r \lambda$$



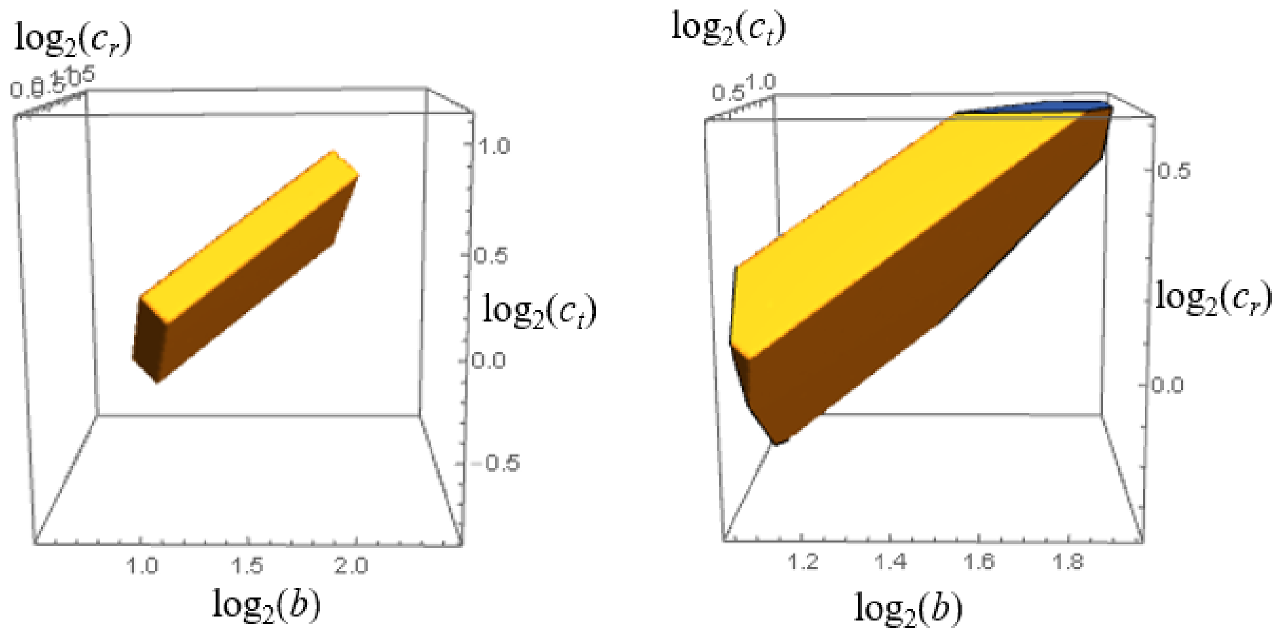
	b	cr	ct	S	AR	λ
AN225	88.4	16.5122	3.96293	905	8.63487	0.24
A380	79.75	17.6594	3.53187	845	7.5267	0.2
A320	34.09	5.99394	1.19879	122.6	9.47902	0.2
Gulfstream IV	23.7	5.73191	1.71957	88.3	6.36116	0.3
U2	32	4.83333	0.966667	92.8	11.0345	0.2
F-16	9.96	4.47711	1.11928	27.87	3.55944	0.25
Mirage 2000	9.13	8.5537	0.427685	41	2.0331	0.05
Cessna 172	11	1.59214	1.35332	16.2	7.46914	0.85
Max value	88.4	17.6594	3.96293	905	11.0345	0.85
Min value	9.13	1.59214	0.427685	16.2	2.0331	0.05

Design space of alternative parameters, with log-axis





Design space of SVD- parameters, with log-axis



Design Space Volumes

Parameter set	Design space volume
b, cr, ct	0.914
S,AR,l	0.877
SVD	0.25

$$H_w = H_1 - H_2 = \log_2 \frac{S_1 / s}{S_2 / s} = \log_2 \frac{S_1}{S_2} = \log_2 \frac{0.914}{0.25} = 1.87\text{bits}$$

Conclusions

- **Design information entropy** represents a measure of the precision by which *a design* is defined relative to the ***design space*** in consideration. It is also proportional to the dimensionality of the design problem.
- Design information entropy can be used as one measure of **complexity**.
- "Thinking outside the box" is the task of finding useful directions to expand the design space.
- **Analytical parametrization** through SVD of a design can be made using sample designs to span the design space. It can also be used to produce scaling models of components. In some sense it can be regarded as the ***ideal parameter set***.